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Distributions of self-interactions and voids in (1+1)-dimensional directed percolation

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We investigate the scaling of self-interactions and voids in $(1+1)$-dimensional directed percolation clusters and backbones. We verify that the meandering of the backbone scales like the directed cluster. A geometric relation between the size distribution and the fractal dimensions of a set of objects is applied to find the scaling properties of self-interactions in directed percolation. Lastly we connect the geometric properties of the backbone with the avalanche distribution generated by interface dynamics at the depinning transition.

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Directed percolation (DP) is the common name for branching processes with an absorbing (here, "dead") state. Denoting the active sites as "live," directed percolation can then be defined as a time-directed process for the propagation of live sites. For a given lattice, each site can come alive with a probability p, if and only if one of its neighbors was live at the previous time step [1]. Directed percolation has a wide applicability, and was proposed to be closely related to Reggeon field theory [2] which describes the evolution of a density ψ of live sites for a large class of branching processes:

$$\frac{d\psi}{dt}=\psi+\frac{d^2\psi}{dx^2}-\lambda\psi^2+\eta(x,t). \tag{1}$$

This is a nonlinear Langevin equation where the noise term obeys $\langle\eta(x,t)\eta(x',t')\rangle\propto\psi(x,t)\delta(x-x')\delta(t-t')$ and thus vanishes when $\psi\rightarrow 0$. Directed percolation has been related to spatiotemporal intermittency [3], self-organized critical models [4–6], and the propagation of interfaces at depinning transitions [4,7–11], as well.

Despite its usefulness, DP remains unsolved. Only in four (spatial) dimensions or more is it tractable, since there the active branches effectively do not meet [12]. In that limit, where we effectively have a random neighbor updating, the critical branching process has a size distribution for extinction at time t that reflects the first-return scaling for a random walk in the number of active sites, $p(t)\propto t^{-3/2}$ [13]. In three dimensions or less, the overlapping of different branches leads to analytical difficulties, and this effect is most pronounced in $(1+1)$-dimensional DP, which we make the focus of this work.

The DP *backbone* is the time-reversal invariant subset of the (time-) directed percolation process. A picture of the $(1+1)$-dimensional DP network consists of branches that die ("dangling ends") and branches that continue, on all scales; within the branches there are voids, of all sizes, enclosed by live branches (see Fig. 1). The backbone, however, has no dangling ends—it has the geometry of a badly tangled fishnet, where stringy lengths separate the multiply connected blobs [14]. Recently much discussion has centered on the backbone in connection with models of interface pinning in disordered media [7,4]. In this paper we study the distribution of self-interactions and voids in DP, in the backbone network, and associate voids with the avalanches expected in interface motion at the depinning threshold [15,9–11]. Our first step is to introduce a general formula that relates the fractal dimensions of a set of objects to the size distribution of the objects.

Size distributions and fractal dimensions. Consider a set of self-similar objects, each with fractal dimension D whose union is a fractal of dimension $D_{\rm tot}$. Define a subset of this total set, that consists of one point from each object. If this point set has dimension $D_{\rm num}$, then the number of objects between sizes s and $s+ds$ contained within a box of length L is

$$n_L(s)ds=L^{D_{\rm num}}s^{-\tau}f(s/L^D)ds, \tag{2}$$

where the scaling function $f(x)$ approaches 1 for $x\ll 1$, and 0 for $x\gg 1$. Since $\int sn_L(s)ds\propto L^{D_{\rm tot}}$, by matching powers of L it follows that the number of objects of size s (for large, fixed L) is distributed as $n(s)\propto s^{-\tau}$ with

$$\tau-1=\frac{D_{\rm num}-D_{\rm tot}+D}{D}. \tag{3}$$

In the case of $D_{\rm num}=D_{\rm tot}$, then the exponent $\tau\geq 2$ can be deduced from $D_{\rm num}$ and D by accounting for the mass of objects that cross the *boundary* of a box of linear dimension L. One obtains:

$$\tau-1=\frac{D_{\rm num}}{D}. \tag{4}$$

We call these respectively the *triplex* and *duplex* formulas. The formulas apply to self-affine objects as long as the dimensions are measured along a common axis, and the objects are disjoint.

One simple application is the distribution of intervals separated by a fractal dust of dimension $D_{\rm ust}$ in one dimension. Each interval can be associated with a point of the dust, so that $D_{\rm num}=D_{\rm ust}$, and is self-similar with the full line, so that $D=D_{\rm tot}=1$. Therefore the interval lengths created by the fractal dust are distributed like

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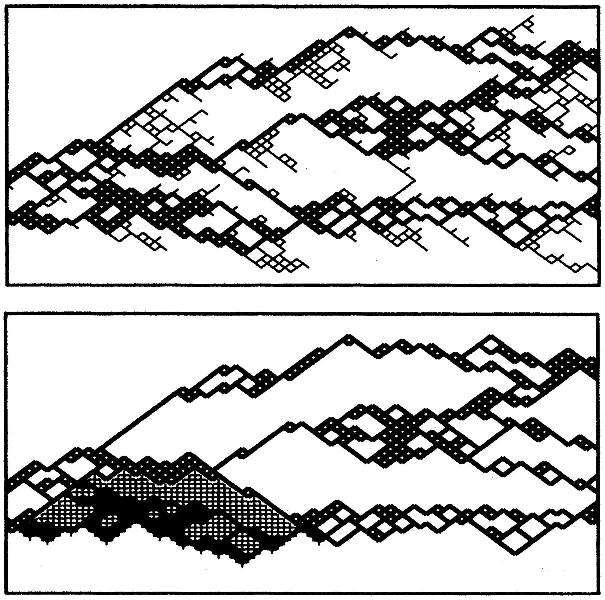

FIG. 1. Upper figure shows directed percolation cluster (thin lines) and backbone (heavy lines). Lower figure shows a punctuation of the backbone hull. The crossed area indicates the erosion due to one punctuation event.

$$n(l)\propto l^{-1-D_{\text{num}}}. \qquad (5)$$

This result is discussed in [16] and was used by [17] for first-return time of activity at a given site.

Directed-percolation exponents. If the percolation parameter p lies below a critical threshold p_c, the propagation of live sites has a finite lifetime. If p lies above p_c, the propagation of live sites can continue forever. For p just below p_c, the timelike correlation length (lifetime) diverges $\propto(p_c-p)^{-\nu_\parallel}$ and the spacelike correlation length (width) $\propto(p_c-p)^{-\nu_\perp}$, where the $(1+1)$-dimensional exponents are $\nu_\parallel=1.733$ and $\nu_\perp=1.097$ [12]. The *order-parameter* exponent β is defined as the scaling of the density of the infinite cluster (above threshold) with distance to threshold ϵ $(=|p-p_c|)$: $\rho\propto\epsilon^{\beta}$. For $(1+1)$-dimensional DP, β=0.277. These three exponents are believed necessary and sufficient to completely characterize DP structures and correlations.

For the backbone network, $\nu_\perp$ and $\nu_\parallel$ are the same as for the full cluster. This was checked by direct simulation of the meandering of the backbone $\langle x^2\rangle\propto t^{1.26\pm0.01}$, i.e., $2\nu_\perp/\nu_\parallel=1.26$, and by the scaling of the number of singly connected bonds ("red bonds" [18]) along the backbone, which leads to $1/\nu_\parallel=0.58$ (see later discussion). By contrast, the order-parameter scaling for the backbone is given by $\tilde{\beta}=2\beta$, because a point belongs to the backbone precisely when it belongs *both* to the forward and the backward DP network [19,20]. (Hereafter, a *tilde* denotes a backbone exponent.) Other exponents are easily deduced, for example, $\chi=\nu_\perp/\nu_\parallel=0.633$, the exponent for the time development of the average width of living sites. The scaling of the mass m of the infinite cluster up to a correlation length $l_\parallel=\epsilon^{-\nu_\parallel}$, $m\propto\epsilon^{\beta-\nu_\parallel-\nu_\perp}$, leads to dimensions for one-dimensional transverse and longitudinal cuts: $D_\perp=1-\beta/\nu_\perp$, $D_\parallel=1-\beta/\nu_\parallel$. In this notation, the full dimension measured longitudinally is then $D_\parallel+\chi$. Measuring how the cluster mass scales with its length for both DP and backbone of directed percolation (BDP), see Figs. 2(b) and 2(c), we get $m(t)\propto t^{1.47\pm0.02}$ and $\tilde{m}(t)\propto t^{1.30\pm0.03}$, and we deduce the respective dimensions of longitudinal cuts $D_\parallel=0.84\pm0.02$ and $\tilde{D}_\parallel=0.67\pm0.03$, consistent with $\tilde{\beta}=2\beta=0.55$. The probability for distance t between subsequent live sites in a longitudinal cut is distributed $\propto t^{-1-D_\parallel}$.

Now if clusters are initiated everywhere in space and time, their size distribution follows from the duplex formula [Eq. (4)] with $D=D_\parallel+\chi$, $D_{\text{num}}=D_{\text{tot}}=1+\chi$:

$$\tau_{\text{all}}-1=\frac{1+\chi}{D_\parallel+\chi}=\frac{\nu_\parallel+\nu_\perp}{\nu_\parallel+\nu_\perp-\beta}. \qquad (6)$$

The exponent for clusters initiated in a single point is then $\tau_{\text{1pt}}=\tau_{\text{all}}-1$.

Self-interactions in directed percolation. To calculate when two branches in $(1+1)$-dimensional directed percolation eventually interfere with each other, we examine voids. A void is a region completely enclosed by the DP network. The size of a cluster void is the number of noncluster sites enclosed by two merging branches; backbone voids are defined similarly. In both cases, the scaling of the void-size distribution can be obtained from the triplex formula. The voids themselves scale like the area of the affine region, $D=1+\chi$. As the voids are dense on the DP network, $D_{\text{num}}=D_\parallel+\chi$ and $D_{\text{tot}}=D=1+\chi$. The void areas s are then power-law distributed $n(s)\propto s^{-\tau_{\text{void}}}$ with

$$\tau_{\text{void}}-1=\frac{D_\parallel+\chi}{1+\chi}=1-\frac{\beta}{\nu_\parallel+\nu_\perp}. \qquad (7)$$

The above argument applied to the backbone voids (replacing $D_\parallel$ with $\tilde{D}_\parallel$) gives $\tilde{\tau}_{\text{void}}=(\nu_\parallel+\nu_\perp-2\beta)/(\nu_\parallel+\nu_\perp)$. These are in agreement with the numerical values $\tau_{\text{void}}=1.93\pm0.02$ and $\tilde{\tau}_{\text{void}}=1.82\pm0.02$ obtained from simulations; see Fig. 2(a). Notice a duality between the distribution of all clusters [Eq. (6)] and the distribution of voids in one cluster: $(\tau_{\text{all}}-1)$ = $(\tau_{\text{void}}-1)^{-1}$. Simply related to τ_{void} is the τ exponent for void length ($\parallel$): $\tau_l-1=(\tau_{\text{void}}-1)(1+\chi)$. This confirms the measured distribution of times between subsequent self-interactions $P(t)\propto t^{-2.55\pm0.02}$. The time between self-interactions differs from the time between subsequent activity at a given site, the latter having a τ exponent of $D_\parallel+1$ (cf. the fractal dust result).

As an application, consider the distribution of voids touching an interface which forms the one-dimensional boundary along the backbone. The distribution of border voids will differ from the overall distribution of voids because larger voids will have a larger probability to touch the one-dimensional (1D) interface. Along a 1D path on the backbone, small voids sit densely, implying $D_{\text{num}}=1$; as before, $D=D_{\text{tot}}=1+\chi$. Thus the distribution of backbone voids along the interface scales with exponent

$$\tilde{\tau}_{\text{1D}}-1=\frac{1}{1+\chi}. \qquad (8)$$

These border voids play a special role in the dynamics of interfaces driven through quenched-disordered media. Examples of such models are the depinning dynamics at an

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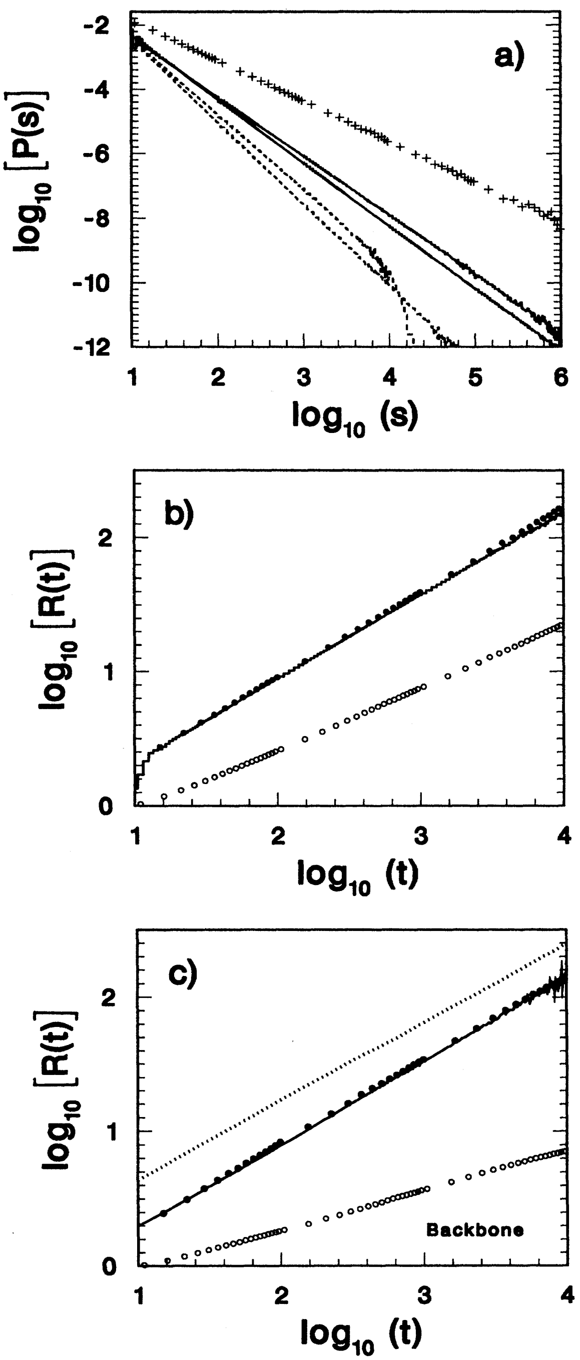

FIG. 2. (a) The dashed lines are the distribution of times between self-interactions. Full lines are the distributions of void areas. In both cases the lower curve represents DP; the upper the backbone. Crosses show the distribution of eroded areas caused by a single punctuation event. The distributions are all measured at critical $p=0.7055$ with clusters of length $l=100\ 000$ for DP and $l=20\ 000$ for the backbone simulations. (b) Solid line is the scaling of average void width with length l along the time t axis. Solid circles show the rms meandering of the DP network. In both cases one sees scaling $\propto t^{0.63\pm0.01}$. Open circles show mass of the network at cross sections perpendicular to the time direction. (c) As in (b) but for BDP instead. The additional dotted line shows the accumulated number of red bonds.

externally tuned critical threshold [7] or when driven similar to invasion percolation [4,8]. In the model of Ref. [4] an interface with small slopes advances at the point of globally minimal pinning, followed by neighboring advances which keep slopes small. This mimics the quasistatic dynamics of the Kardar-Parisi-Zhang (KPZ) equation with quenched noise [10]. This intermittently jumping interface gets pinned along a DP backbone of high pinning strengths [9]. The advance of the interface occurs in bursts associated with punctuations of voids in the underlying backbone [8,15,9–11]. We now associate the distribution of these bursts with the distribution of voids on the outer surface (hull) of the DP backbone.

A void of length l that borders the backbone hull has a probability to be punctuated proportional to the number of singly connected sites $n_{\rm red}(l)$ on the length l of the hull. These are the famous *red* bonds introduced by Stanley [18]. To calculate $n_{\rm red}(l)$ consider a segment of length l on the backbone hull at the critical point $p=p_c$. If p is decreased by an amount ϵ below the critical point, the cluster may be disconnected on length l because each site on the cluster has probability ϵ to be removed. The average number of bonds that disconnect on length l is then $N(l,\epsilon)=n_{\rm red}(l)\epsilon+O(\epsilon^2)$. Ignoring the higher-order terms, the correlation length of the cluster becomes equal to l at the ϵ value where $N(l,\epsilon)$ is equal to 1. From the known correlation length $l=\epsilon^{-\nu_\parallel}$ we get $n_{\rm red}(l)=l^{1/\nu_\parallel}$. This result was obtained for isotropic percolation by Coniglio [21]. This is confirmed by direct numerical simulation of the number of singly connected bonds on length l: $n_{\rm red}\propto l^{0.58\pm0.02}$. Weighting the void-size distribution by the chance a given void is punctuated $[\propto n_{\rm red}(l)]$ and noting that the void area is $s=l^{1+\chi}$, one obtains that the distribution of the areas of the interface avalanches is a power law with exponent

$$\tilde{\tau}_{\rm wtd}-1=\frac{1}{1+\chi}-\frac{1}{\nu_\parallel}\frac{1}{1+\chi}=\frac{\nu_\parallel-1}{\nu_\parallel+\nu_\perp}, \tag{9}$$

which is identical to the earlier formula of Maslov and Paczuski [11], and also discussed for other interface models by [22]. Inserting the DP values $\nu_\perp=1.10$ and $\nu_\parallel=1.73$ the obtained $\tau=1.259\pm0.005$ agrees both with large-scale simulations of the interface model [4] ($\tau=1.255\pm0.005$ [23]), as well as with direct simulations of eroded void areas resulting from the removal of one backbone hull site [8]. However, a punctuation typically leads to the elimination of an entire cascade of backbone voids, see Fig. 1(b) (the probability to eliminate N voids with one punctuation is in fact $\propto N^{-1.35\pm0.05}$ [24]). Therefore the agreement between numerics and the above formula for τ, using DP exponents, may only be approximative. In fact, large-scale simulations [23] show that $\nu_\perp(\text{model})=1.05\pm0.02$ for the interface model [4] is smaller than $\nu_\perp=1.10$ for DP. The measured $\tau=1.255\pm0.005$ [23] and Eq. 9 then dictate a new value of $\nu_\parallel(\text{model})$ and a slightly modified value of the interface roughness exponent $\chi=\nu_\perp/\nu_\parallel$ [25].

The interface dynamics may be considered as a way to assign weights to branches of the DP backbone. The observation of new exponents of DP with weighted branches indicate that directed percolation could exhibit multiscaling. This phenomenon might also explain the numerical observation by [26] of close but not identical DP exponents (τ,D) for the "evolution" model [5].

Finally, the above derivation of the avalanche exponent can be repeated for invasion percolation [27,28] using the triplex formula. Identify D_{num} as the dimension of the region of the growing cluster where new bursts can initiate ($D_{\mathrm{num}}=D_{\mathrm{perimeter}}$ which depends on the invasion rule) and D as the dimension of a single avalanche (cluster). The ratio D_{num}/D is the exponent for border clusters, defined as clusters that have at least one connection to the invading cluster. The exponent τ_{inv} for the invasion avalanches is given by weighting these border clusters according to the number of bonds through which they can be invaded, i.e., the red bonds on the scale defined by the cluster size. Thus

$$\tau_{\mathrm{inv}}-1=\frac{D_{\mathrm{num}}}{D}-\frac{1/\nu}{D}. \tag{10}$$

This relation was derived in [29].

Summary. We have investigated self-interactions and voids in both DP clusters and backbones, and have discussed how their scaling depends on the DP exponents. It is noteworthy that the self-interaction time for branches has finite mean, in contrast to the time between subsequent activity at a given site. We have also discussed avalanches occurring in models of interface motion near the depinning threshold in terms of voids of the DP backbone. Further, we presented a general formula for relating dimensions to size distributions which opens up for a study of scaling properties of many other self-affine and self-similar structures.

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[24] The eroded area resulting from a single punctuation consists of a cascade of backbone voids. The number of voids in this cascade is distributed $n(N_{\mathrm{void}})\sim N^{-1.35\pm0.05}$ (numerics). As voids sit densely on the backbone, the number of voids is proportional to the backbone mass $\tilde{m}$ on the scale of the eroded area s: $N_{\mathrm{void}}\sim\tilde{m}\sim s^{(\tilde{D}_\parallel+\chi)/(1+\chi)}$. Transforming the void area distribution (i.e., $\tilde{\tau}_{\mathrm{wtd}}$) to void number N_{void} distribution: $n(N_{\mathrm{void}})\sim N_{\mathrm{void}}^{-\tilde{\tau}_{\mathrm{casc}}}$, where $\tilde{\tau}_{\mathrm{casc}}-1=1-D_{\mathrm{red}}/\tilde{D}_\parallel+\chi=\nu_\parallel-1/\nu_\parallel+\nu_\perp-2\beta$.
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